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Dynamics and sound emission of a spherical cavitation bubble in a dilute polymer solution

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The effect of polymer additive on the growth and collapse of a spherical vapour bubble is investigated theoretically, under conditions appropriate for cavitation (negligible influence of heat transfer, Newtonian viscosity, etc.). The polymer-induced stress is calculated using the yo-yo model of the polymer dynamics in transient extensional flows (Ryskin 1987*a*). The resulting equation of bubble dynamics is solved numerically; an approximate analytical solution is also obtained. It is found that the growth of a bubble is not affected by the polymer, but the final stage of the collapse is. After following closely the classical inviscid-fluid solution, the collapse is abruptly arrested, and the bubble wall velocity is reduced to nearly zero. The peak acoustic pressure of the radiated sound is also reduced, and the high-frequency part of the acoustic spectrum is sharply curtailed.

1. Introduction

The effect of polymer additives on cavitation has been studied for about 20 years, beginning with the papers of Ellis & Hoyt (1968), Ellis, Waugh & Ting (1970) and Fogler & Goddard (1970). The experiments of Ellis & Hoyt (1968) and Ellis *et al.* (1970) demonstrated that addition of minute amounts of flexible long-chain polymers, of drag-reducing type, could significantly suppress cavitation, i.e. reduce the incipient cavitation number by a factor of 2 to 3, in high-speed flows past hemispherically nosed bodies. Fogler & Goddard (1970) analysed theoretically the collapse of a spherical bubble in a linear viscoelastic (Maxwell) fluid and concluded that elasticity of the fluid could significantly retard the collapse, the bubble motion becoming oscillatory in character, i.e. consisting of a series of collapses and rebounds, modified by viscous damping.

Further work showed that polymer additives also suppress cavitation in a jet flow (Hoyt 1976; Hoyt & Taylor 1981), in a single-vortex flow (Hoyt 1978), and in a rotating-disc flow (Ting 1978); they also suppress acoustic cavitation (Hoyt 1977; Crum & Brosey 1984) and strongly influence cavitation damage and noise (Ashworth & Procter 1975; Shima *et al.* 1985; Nanjo, Shima & Tsujino 1986; Oba, Ito & Uranishi 1978; Reitzer, Gebel & Scrivener 1985; Tsujino 1987). However, the connection between these phenomena and the effect of polymer on the dynamics of a single bubble remained elusive.

The experimental work is severely complicated by the fact that the polymer additive also changes the overall flow field and thus creates different conditions for cavitation. In fact, it was shown by van der Muelen in 1976, and independently by Gates in 1977 (see Arakeri & Acosta 1981 for references and discussion), that the suppression of cavitation in flows past hemispherically nosed bodies is due to this last effect, viz. the polymer alters the structure of the boundary layer, leading to an earlier transition to turbulence and elimination of laminar separation. This delays the cavitation inception since the region of reattachment of the separated laminar boundary layer is the critical zone for cavitation inception due to intense pressure fluctuations there (Arakeri & Acosta 1981).

Furthermore, the experiments with individual bubbles (Ting & Ellis 1974; Chahine & Fruman 1979; Kezios & Schowalter 1986) showed no effect of polymer on the dynamics of a spherical bubble, while the theoretical analyses of Ting (1975, 1977), who used the Oldroyd constitutive relation, and of Hara & Schowalter (1984), who used the Maxwell one but in convected (Lagrangian) coordinates, predicted that the polymer effect will be negligible for the values of the parameters (Deborah number, etc.) that could be encountered in cavitation. (In fact, the results of Hara & Schowalter 1984 show no effect of rheology on the collapse of a spherical bubble for the values of Deborah number up to 10^5.)

At the moment, the consensus appears to be that the dynamics of a spherical cavitation bubble is practically unaffected by polymer additive, and that the strong influence of polymer on the cavitation phenomena is due to the alteration of the overall flow field, as well as to the polymer effect on the dynamics of non-spherical bubbles (Chahine 1982; Kezios & Schowalter 1986).

The purpose of the present work is to investigate the polymer effect on the dynamics of a spherical cavitation bubble using the yo-yo model of polymer dynamics (Ryskin 1987*a*). In brief, the yo-yo model says that a polymer chain, subjected to an elongational flow with a strain rate above, approximately, the inverse of the polymer relaxation time, will begin to *unravel*; soon thereafter within the unravelling chain at least one large chain segment will be stretched taut. This segment will then remain taut and continue to grow at the expense of the adjoining, still coiled portions of the chain at its ends; the latter portions will be moving apart with the flow, simultaneously diminishing in size. If the flow later becomes weak, the chain will curl back into a coil. The taut central portion generates large additional stress via a dissipative mechanism.

It is important to keep in mind that the yo-yo model is not intended to describe the polymer dynamics in detail. Indeed, one can easily visualize some likely features of the actual dynamics which are not reflected in the yo-yo model (e.g. several segments of the chain may be unravelling in parallel, etc.). However, the yo-yo model is so constructed that these details of the polymer dynamics become unimportant (Ryskin 1987*a*; see especially pp. 426–428, 438).

The model has been successful in explaining some laminar flow experiments (Ryskin 1987*a*), as well as the turbulent drag reduction phenomenon (Ryskin 1987*b*). Of course, more flow situations will have to be analysed and compared with experiments before the validity (or invalidity) of the yo-yo model can be established.

Most of the works concerned with fluid dynamics of non-Newtonian fluids utilize, in addition to the mass and momentum conservation laws, constitutive equations, which allow one, in principle, to calculate the stress, given the history of the strain rate in a fluid element. It is rather obvious that none of the existing constitutive equations could describe the rheology of a solution in which polymer molecules behave according to the yo-yo model, owing to the non-smooth (on–off) character of the polymer contribution to the stress (see Ryskin 1987*a*). A new general constitutive 'rule' could be derived on the basis of the yo-yo model (plus, perhaps, some additional assumptions), though this rule would certainly not have the form of a differential equation. One could then use this rule to solve fluid dynamics problems, including the one at hand.

However, the problem considered here is such that it can be solved using the molecular model directly. Such a 'molecular' approach, linking directly molecular dynamics on the microscale with fluid dynamics on the macroscale, offers two benefits: first, the physics of the problem is made quite clear; second, only those assumptions are made, in addition to the yo-yo model itself, which are absolutely necessary for advancing the solution.

What follows is an application of this approach to the problem of cavitation bubble dynamics.

2. The general equation of bubble dynamics

The basic equation of bubble dynamics in an incompressible non-Newtonian fluid has been derived by Fogler & Goddard (1970), following the previous derivations for an inviscid fluid by Besant (1859) and by Rayleigh (1917) and for a Newtonian fluid by Plesset (see Plesset & Prosperetti 1977). This basic equation is

$$\rho(R\ddot{R}+\tfrac{3}{2}\dot{R}^2) = p_i - p_{\mathrm{amb}} - \frac{2\sigma}{R} + 3\int_R^\infty \frac{\tau_{rr}}{r}\,\mathrm{d}r,$$

where R is the radius of the bubble, a dot means differentiation with respect to time, ρ is the density of the liquid, p_i and p_{amb} are the pressures inside the bubble and in the liquid far from the bubble, respectively, σ is the surface tension, r is the radial coordinate, and τ_{rr} is the indicated component of the deviatoric stress tensor $\boldsymbol{\tau}$; the full stress tensor is given by $-p\boldsymbol{I}+\boldsymbol{\tau}$ and $\mathrm{Tr}(\boldsymbol{\tau}) = 0$.

In what follows we assume that both p_{amb} and p_i are constant. The constancy of p_i is a good approximation for the case of a bubble filled with vapour only (i.e. no non-condensable gas), at least until the velocity $\dot{R}$ of the wall of the collapsing bubble becomes comparable with the speed of sound in the vapour, in which case the vapour does not have enough time to condense, and its pressure begins to rise (see Plesset & Prosperetti 1977). Thus in the present work $p_i = p_v$, where the equilibrium vapour pressure p_v is a function of temperature only; e.g. for water at 15 °C, $p_v = 1.7\times10^4$ dyn/cm$^2 \approx 0.017$ atm.

Condensation of the vapour (during collapse), or evaporation of the liquid (during growth), that must occur in order to keep the vapour pressure inside the bubble at its equilibrium value, entail heat transfer since the latent heat associated with the phase transition must be removed or delivered. This heat transfer may influence the bubble dynamics very strongly, and is, in fact, a controlling factor in boiling. However, its importance depends crucially on the equilibrium vapour density, and when the latter is relatively small ('cold' liquid), the thermal effects can be neglected. Such is the case for a cavitation bubble (Plesset & Prosperetti 1977); we thus neglect thermal effects in the present work.

Consider now the surface-tension effect. The magnitude of the surface tension is certainly important in determining whether a given nucleus begins to grow: for given $p_v - p_{\mathrm{amb}}$, the nucleus radius must be greater than the critical value R_c such that

$$p_v - p_{\mathrm{amb}} - \frac{2\sigma}{R_c} = 0.$$

However, if the growth begins, the influence of the surface tension term quickly diminishes owing to its inverse dependence on radius; if the initial radius was substantially greater than R_c, the influence of the surface tension is negligible from the very beginning.

In the case of a collapsing bubble, the surface tension term is typically small compared to $p_{amb}-p_v$ at the initial stages of the collapse since R is relatively large then: note that for $p_{amb}-p_v = 1$ atm and $\sigma = 70$ dyn/cm, the bubble radius must be $O(1\ \mu m)$ in order for the surface-tension term to be comparable with the pressure-difference term. By the time the radius of the collapsing bubble gets so small, other assumptions (stated above or implied) are likely to break down: the velocity $\dot{R}$ may become comparable with the speed of sound in the vapour, so p_i is no longer equal to p_v; it may further become comparable with the speed of sound in the liquid, in which case the latter cannot be considered incompressible; finally, the assumption of the spherically symmetric motion becomes invalid owing to shape instabilities (Plesset & Prosperetti 1977). Moreover, both the pressure-difference term and the surface-tension term become negligibly small in comparison with the inertial terms at this stage of the collapse (see §5).

It is clear then that in many realistic situations the surface-tension effects are likely to be unimportant. In order to concentrate on the main topic of the present work – the influence of polymer additive – we shall neglect the surface-tension effects here.

The basic equation then takes the form

$$\rho(R\ddot{R}+\tfrac{3}{2}\dot{R}^2) = p_v - p_{amb} + 3\int_R^\infty \frac{\tau_{rr}}{r}\,dr. \tag{2.1}$$

The effect of rheology of the liquid is described by the last term in (2.1). In the absence of polymer, i.e. in the Newtonian solvent alone, this term reduces to

$$-4\eta_s\frac{\dot{R}}{R}, \tag{2.2}$$

where η_s is the solvent viscosity.

The Newtonian viscous effect described by (2.2) is normally negligible in the dynamics of cavitation bubbles (see Plesset & Prosperetti 1977; see also the discussion near the end of §5 of the present paper). Thus, if we find below that in a particular motion the effect of polymer additive is comparable with, or less than, the purely viscous effect due to the solvent alone, we shall consider the polymer additive effect to be unimportant.

3. Polymer effect in an extensional flow

Here we recall what the yo-yo model predicts for the polymer effect in an extensional flow with given kinematics. The dynamics of a spherical bubble involves only two types of flow, both having spherical symmetry: the diverging (point source) flow in the case of growth, and the converging (point sink) flow in the case of collapse. In the Lagrangian frame of a fluid particle the velocity field is an extensional flow, with the rate-of-strain tensor

$$\boldsymbol{E} = E\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix},$$

where the principal direction 1 is along the streamline. The principal strain rate $E_{11} \equiv E_{rr} \equiv E$ is given by

$$E = -2\frac{R^2\dot{R}}{r^3}. \tag{3.1}$$

A rheologist would say that uniaxial extension ($E > 0$) obtains in collapse, while biaxial extension ($E < 0$) obtains in growth. In both cases, however, the symmetry of the rate-of-strain tensor remains the same, namely, the tensorial structure of $\boldsymbol{E}$ is that of a uniaxial deviator. (The meaning of the term 'uniaxial' in rheology is, obviously, narrower.) That is, $\boldsymbol{E}$ defines a single direction in space (this would not be the case if all three principal values of $\boldsymbol{E}$ were different, as they are, for example, in a two-dimensional straining motion, i.e. a hyperbolic flow). This single direction must then determine the structure of the additional deviatoric stress due to polymer molecules $\boldsymbol{\tau}'$. This means that (when $\boldsymbol{E}$ has a tensorial structure of a uniaxial deviator) $\boldsymbol{\tau}'$ may be written

$$\boldsymbol{\tau}' = 2\zeta\eta_s \boldsymbol{E},$$

where ζ is the ratio of the contributions to the bulk deviatoric stress due to the presence of the macromolecules and due to the pure solvent. The combined deviatoric stress due to the Newtonian solvent and the polymer molecules together is then

$$\boldsymbol{\tau} = 2\eta_s \boldsymbol{E} + \boldsymbol{\tau}' = 2(1+\zeta)\,\eta_s \boldsymbol{E}.$$

The value of ζ is, however, determined somewhat differently in two cases $E > 0$ and $E < 0$. Let us denote ζ in the former case as ζ_+ and in the latter as ζ_-. The former case was considered in detail in Ryskin (1987*a*); the result is

$$\zeta_+ = \kappa\xi^3,$$

where $\kappa \approx 0.3c[\eta]$, c is the polymer concentration by weight, and $[\eta]$ is the intrinsic viscosity. The numerical factor 0.3 is actually a very weak (logarithmic) function of $c[\eta]$, see equation (5) of Ryskin (1987*a*).

The variable ξ is the relative elongation of a material line element directed along the stretched polymer chains, from the point where the supercritical unravelling of the macromolecules began. This critical point is defined by the condition that there $E = E_{cs}$, where the critical strain rate E_{cs} is related to the longest relaxation time of the macromolecule τ_1 via the coil–stretch criterion

$$E_{cs}\tau_1 = O(1).$$

The general evolution equation for a material line element is well known (see e.g. Batchelor 1967, p. 132), but will not be needed here since the element in question is always lying along the radial direction 1 and so ξ is easily calculated (see (5.4) below).

Consider now the $E < 0$ case. This case is significantly more complicated. Unlike in the $E > 0$ case, where it was clear that the polymer chains are stretched along the principal direction 1, here all the directions of stretching in the plane normal to 1 are equally probable. Exactly how this symmetry is broken, and whether this may lead to the violation of the assumed spherical symmetry of the kinematics, is beyond the scope of this paper. We shall simply assume here that in different fluid elements the directions of stretching may be different, and that the resulting stress $\boldsymbol{\tau}'$ is the average of the stress tensors that would result from stretching in all these *possible* directions with equal probability. Then $\boldsymbol{\tau}'$ will have the symmetry of a uniaxial deviator and thus can be found as the half-sum of the stress tensors that would result from stretching in any two mutually perpendicular directions among the above possible ones. These two directions can be taken, without loss of generality, as the principal directions 2 and 3 of $\boldsymbol{E}$. The general expression for the polymer-induced stress is (see Batchelor 1971 and Ryskin 1987*a*)

$$\boldsymbol{\tau}' = 3\kappa\xi^3\eta_s \boldsymbol{n}\cdot\boldsymbol{E}\cdot\boldsymbol{n}(\boldsymbol{n}\otimes\boldsymbol{n}-\tfrac{1}{3}\boldsymbol{I}),$$

where $\boldsymbol{n}$ is a unit vector directed along the stretched polymer chains.

We thus write, for a given relative elongation ξ,

$$\boldsymbol{\tau}' = \tfrac{1}{2}(\boldsymbol{\tau}'_{(2)}+\boldsymbol{\tau}'_{(3)}),$$

$$\boldsymbol{\tau}'_{(2)} = 2\kappa\xi^3\eta_s E_{22}\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix},$$

$$\boldsymbol{\tau}'_{(3)} = 2\kappa\xi^3\eta_s E_{33}\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $$E_{22} = E_{33} = -\tfrac{1}{2}E.$$

The result is $$\boldsymbol{\tau}' = \tfrac{1}{2}\kappa\xi^3\eta_s \boldsymbol{E},$$

which means that $$\zeta_- = \tfrac{1}{4}\kappa\xi^3.$$

The variable ξ is here the relative elongation of a material line element lying in the plane normal to the direction 1, from the critical point where $E = -2E_{cs}$.

We are now in a position to calculate the stress for both the growth and the collapse phases of bubble dynamics. But first let us summarize the governing equations.

The basic equation of bubble dynamics is (equation (2.1))

$$\rho(R\ddot{R}+\tfrac{3}{2}\dot{R}^2) = p_v - p_{amb} + 3\int_R^\infty \frac{\tau_{rr}}{r}\,dr.$$

The initial conditions, in both the growth and the collapse cases, are

$$R = R_0;\quad \dot{R} = 0 \quad \text{at} \quad t = 0.$$

Growth will occur if $p_{amb} < p_v$, and collapse if $p_{amb} > p_v$.

As mentioned earlier, the Newtonian viscous stress has negligible influence on the dynamics of cavitation bubbles. Therefore, its contribution to τ_{rr} will be neglected, and only the contribution due to polymer will be taken into account, i.e.

$$\tau_{rr} = \tau'_{rr} = 2\zeta\eta_s E.$$

We thus have

$$(a)\ \text{in growth:}\quad \tau_{rr} = -\kappa\xi^3\eta_s\frac{R^2\dot{R}}{r^3},$$

where ξ is the relative elongation of a material line element normal to the radial direction, from the critical point where $E = -2E_{cs}$;

$$(b)\ \text{in collapse:}\quad \tau_{rr} = -4\kappa\xi^3\eta_s\frac{R^2\dot{R}}{r^3},$$

where ξ is the relative elongation of a material line element lying along the radial direction, from the critical point where $E = E_{\text{cs}}$.

The growth and the collapse of a cavitation bubble are analysed below, in this order (chosen to reflect the 'life cycle' of the bubble). The reader should be warned, however, that the main result in the case of the growth is the prediction of no observable influence of the polymer additives on dynamics (at any rate, no more than that of the Newtonian viscosity). Some readers may thus choose to skip the section on growth altogether.

4. The growth of a bubble

Let us first recall the solution of this problem in the case of a Newtonian fluid. With the viscous effects neglected, the basic equation (2.1) reduces to

$$\rho(R\ddot{R}+\tfrac{3}{2}\dot{R}^2) = p_{\text{v}}-p_{\text{amb}}.$$

This can be integrated once to give (see e.g. Plesset & Prosperetti 1977; here the subscript zero signifies the initial conditions, and the initial velocity of the bubble wall $\dot{R}_0$ is taken to be zero)

$$\dot{R} = \left(\frac{2}{3}\frac{p_{\text{v}}-p_{\text{amb}}}{\rho}\right)^{\frac{1}{2}}\left[1-\left(\frac{R_0}{R}\right)^3\right]^{\frac{1}{2}}. \tag{4.1}$$

That is, after a short initial period during which $R \lesssim 2R_0$, the velocity of the bubble wall is practically constant and equal to its asymptotic value for the growth phase

$$\dot{R}_\infty = \left(\frac{2}{3}\frac{p_{\text{v}}-p_{\text{amb}}}{\rho}\right)^{\frac{1}{2}}. \tag{4.2}$$

For example, for $p_{\text{v}}-p_{\text{amb}} = 1$ bar $= 10^6$ dyn/cm^2 ($= 0.987$ atm) and $\rho = 1$ g/cm^3, $R_\infty = 816$ cm/s.

The principal rr value E of the rate-of-strain tensor is negative; as can be seen from (3.1), at any given moment the maximum absolute value of E is reached at the surface of the bubble

$$|E|_R = 2\frac{\dot{R}}{R}.$$

Clearly, $|E|_R$ as a function of time is 0 at the beginning of the growth, then quickly reaches some maximum value as $\dot{R}$ approaches $\dot{R}_\infty$ while R is still of order R_0, and then enters the asymptotic regime, diminishing with time approximately in inverse proportion to the bubble radius. The maximum value of $|E|$ over the whole growth phase is thus of the order

$$\frac{1}{R_0}\left(\frac{p_{\text{v}}-p_{\text{amb}}}{\rho}\right)^{\frac{1}{2}}.$$

Now, if this maximum value is less than the critical value $2E_{\text{cs}}$, no polymer chains will be stretched. We thus observe that the behaviour of the polymer during the growth of a cavitation bubble should be determined by the dimensionless parameter

$$\alpha \equiv \frac{1}{E_{\text{cs}}R_0}\left(\frac{p_{\text{v}}-p_{\text{amb}}}{\rho}\right)^{\frac{1}{2}}, \tag{4.3}$$

so that for $\alpha \lesssim 1$ the polymer effect is absent, while for $\alpha \gg 1$ it may be important. Note also that

$$\alpha \approx \frac{\tau_1}{R_0}\left(\frac{p_v - p_{\text{amb}}}{\rho}\right)^{\frac{1}{2}}. \tag{4.4}$$

Consider now a bubble growing in a dilute polymer solution, in the case where α is sufficiently large that the coil–stretch (CS) events do occur. The polymer effect on the bubble dynamics will be described by the last term in (2.1), i.e. (neglecting the Newtonian viscous stress)

$$3\int_R^\infty \frac{\tau_{rr}}{r}\,\mathrm{d}r = 6\eta_s \int_R^\infty \frac{\zeta_- E}{r}\,\mathrm{d}r = -3\kappa\eta_s R^2\dot{R}\int_R^\infty \frac{\xi^3}{r^4}\,\mathrm{d}r, \tag{4.5}$$

where ξ is the relative elongation of a material line element, lying in the plane normal to the radial direction, from the critical point where E was equal to $-2E_{\text{cs}}$. Denoting the position of this point as r_{cs}, we have

$$\xi = \frac{r}{r_{\text{cs}}},$$

since as a thin spherical shell of fluid expands, each line element in its tangential plane grows as r (and so its area grows as r^2). Thus the last integral in (4.5) is

$$\int_R^\infty \frac{\xi^3}{r^4}\,\mathrm{d}r = \int_R^\infty \frac{\mathrm{d}r}{r r_{\text{cs}}^3}. \tag{4.6}$$

To evaluate the integral, we need to know r_{cs} as a function of r (the current position of the fluid element); this function is, however, different for different fluid elements, i.e. r_{cs} is a function not only of r but also of time. The task seems formidable at first; it can, however, be accomplished, at the price of introducing some reasonable approximations, as follows.

4.1. *Calculation of the position of the critical point*

Let us label each fluid element by its initial (at time zero) position, denoted r_0. Conservation of the volume of liquid within a spherical material surface passing through this fluid element gives

$$r^3 - R^3 = r_0^3 - R_0^3. \tag{4.7}$$

The rate of strain in this fluid element is then given by

$$E = -\frac{2R^2\dot{R}}{r_0^3 - R_0^3 + R^3}.$$

Note that this is a function of r_0 and time (via R and $\dot{R}$).

Consider first the simplest case of extremely low polymer concentration, so that the influence of the polymer on the kinematics is very small. Then to a first approximation we can use the inviscid-fluid solution (4.1) to find the maximum value of $|E|$ ever experienced by a particular fluid element; if this value is less than $2E_{\text{cs}}$, the CS event never occurs in this fluid element.

We thus need to find the value of R at which the expression

$$\frac{R^2}{r_0^3 - R_0^3 + R^3}\left[1 - \left(\frac{R_0}{R}\right)^3\right]^{\frac{1}{2}}$$

is at maximum. It is easy to show that this value is $R = x^{\frac{1}{3}}R_0$, where x satisfies the quadratic equation

$$2x^2-\left[1+4\left(\frac{r_0}{R_0}\right)^3\right]x+\left(\frac{r_0}{R_0}\right)^3-1=0,$$

and is well approximated by

$$x \approx 2\left(\frac{r_0}{R_0}\right)^3+0.5.$$

Thus the value of R when $|E|$ reaches its maximum is given by

$$R^3 \approx 2r_0^3+0.5R_0^3,$$

and the fluid element labelled r_0, in which this maximum is reached, at this moment is located at r, given by

$$r^3 \approx 3r_0^3-0.5R_0^3 \approx 1.5R^3-1.25R_0^3.$$

Let us now determine r_0^{lim} such that the CS events occur in all fluid elements with labels $r_0 \leqslant r_0^{\text{lim}}$, but do not occur in fluid elements with $r_0 > r_0^{\text{lim}}$. To simplify things a little, we assume that r_0^{lim} is not too close to R_0 (otherwise the polymer chains would stretch only in the immediate vicinity of the bubble and their effect would likely be negligible). As soon as r_0 is about $2R_0$ or larger, the last expression can be rewritten as

$$r^3 \approx 3r_0^3 \approx 1.5R^3. \tag{4.8}$$

Now r_0^{lim} will be determined by the condition that the maximum value of $|E|$ in the fluid element labelled r_0^{lim} is equal to $2E_{\text{cs}}$. For any $r_0 \gtrsim 2R_0$, the maximum value of $|E|$ can be written as (see (3.1) and (4.8))

$$\frac{1.06}{r_0}\left(\frac{2}{3}\frac{p_{\text{v}}-p_{\text{amb}}}{\rho}\right)^{\frac{1}{2}}, \tag{4.9}$$

since by the moment this value is reached, $R > 2.5R_0$, and the bubble wall velocity $\dot{R}$ has practically reached its asymptotic value (4.2).

Thus we obtain

$$\frac{1.06}{r_0^{\text{lim}}}\left(\frac{2}{3}\frac{p_{\text{v}}-p_{\text{amb}}}{\rho}\right)^{\frac{1}{2}} = 2E_{\text{cs}},$$

and so

$$\frac{r_0^{\text{lim}}}{R_0} = 0.43\alpha,$$

this result being reasonably accurate for $\alpha > 5$.

By the definition of r_0^{lim}, the CS event occurs in the fluid element labelled r_0^{lim} at the moment when $|E|$ there reaches its maximum value, and so for this fluid element, by (4.8),

$$r_{\text{cs}} \approx 1.44r_0^{\text{lim}}.$$

In a fluid element with label $r_0 < r_0^{\text{lim}}$ the CS event occurs before this fluid element reaches the position where its strain rate is at maximum: the latter, being inversely proportional to r_0 (see (4.9)), is higher than $2E_{\text{cs}}$, so that the critical value for the CS event is reached earlier. In fact, for large α, r_{cs} for the fluid elements with labels $r_0 \ll r_0^{\text{lim}}$ will be just slightly above r_0.

We shall assume in what follows that $r_{\text{cs}} = r_0$ for all fluid elements where the CS event occurs. This will simplify things greatly, but has an effect of slightly exaggerating the polymer stress, for example, the value of ξ in the fluid element with the label r_0^{lim} becomes 1.44 (instead of 1) at the moment of the true CS event, i.e. the beginning of the supercritical unravelling. The difference, however, is not too large, especially if one takes into account the unavoidable uncertainty in determination of the exact moment of the CS event, as well as of the hydrodynamically effective length of the polymer chain at this moment (see Ryskin 1987*a*).

4.2. *The magnitude of the polymer effect*

Now we can calculate the integral in (4.6), namely

$$\int_R^\infty \frac{\mathrm{d}r}{r r_{\mathrm{cs}}^3} \approx \int_R^{r^{\lim}} \frac{\mathrm{d}r}{r r_0^3} = \int_R^{r^{\lim}} \frac{\mathrm{d}r}{r(r^3 - R^3 + R_0^3)},$$

where $$(r^{\lim})^3 = (r_0^{\lim})^3 - R_0^3 + R^3.$$

Taking into account that our derivation is valid for $R > 2.5R_0$ (note that at earlier moments ξ is certainly less than 2.5 everywhere and so the polymer stress cannot be significant), we obtain the following value for this integral:

$$\frac{1}{R^3}\log\left[\left(\frac{R_0}{R}\right)^3 + \left(\frac{1}{0.43\alpha}\right)^3\right]^{-\frac{1}{3}},$$

and so the term in (2.1) due to the polymer effect is given by (see (4.5))

$$-3\kappa\eta_s\frac{\dot{R}}{R}\log\left[\left(\frac{R_0}{R}\right)^3 + \left(\frac{1}{0.43\alpha}\right)^3\right]^{-\frac{1}{3}}.$$

The ratio of this polymer contribution to the one due to the Newtonian solvent, equation (2.2), is

$$\tfrac{3}{4}\kappa\log\left[\left(\frac{R_0}{R}\right)^3 + \left(\frac{1}{0.43\alpha}\right)^3\right]^{-\frac{1}{3}}.$$

The log-factor is at maximum when $R \to \infty$, so this ratio is not larger than

$$\tfrac{3}{4}\kappa\log 0.43\alpha.$$

An upper limit on α can be estimated by taking $p_v - p_{\mathrm{amb}} = 100$ bar; $R_0 = 1$ μm, and $\tau_1 = 10^{-2}$ s. Then (4.4) gives

$$\alpha \approx 10^6$$

and $\log 0.43\alpha \approx 13$.

On the other hand, $\kappa \approx 0.3c[\eta]$, where $c[\eta]$ must be significantly less than 1 in order for the polymer solution to be dilute ($c[\eta]$ is essentially c/c^*, where c^* is the overlap threshold, see de Gennes 1979). We thus conclude that, somewhat surprisingly, the polymer effect on the dynamics of a growing cavitation bubble is at most of the same order of magnitude as the Newtonian viscous effect, and so can be safely neglected. Note that even though our derivation assumed that the polymer effect on kinematics is small, the above conclusion is independent of this assumption: the polymer can only slow down the bubble growth, thus decreasing the rate-of-strain values everywhere. This will have an effect of delaying the CS events in some fluid elements and eliminating them altogether in others. This means that our results provide an upper limit for the magnitude of the polymer effect.

Qualitatively, the effect of polymer is negligible during the growth of a bubble because in the beginning of the growth, when the strain rates are still high, the polymer chains are only slightly elongated; they do reach high elongations later on, but by that time the strain rates drop. As a result, the polymer-induced stresses are never large.

Experimentally, no influence of polymer additives on bubble growth has ever been observed; reliable observations are very difficult, however.

5. The collapse of a bubble

Let us first recall the classical solution, corresponding to the case of a negligibly small last term in (2.1). We have (see e.g. Plesset & Prosperetti 1977)

$$\dot{R} = -\left(\frac{2}{3}\frac{p_{\mathrm{amb}}-p_{\mathrm{v}}}{\rho}\right)^{\frac{1}{2}}\left[\left(\frac{R_0}{R}\right)^3-1\right]^{\frac{1}{2}}, \tag{5.1}$$

where the subscript '0' means at time zero, which in this case is taken to correspond to the beginning of collapse. The bubble wall velocity $\dot{R}\to\infty$ as $R\to 0$; nevertheless, the 'inertial' velocity scale can be identified as

$$U \equiv \left(\frac{p_{\mathrm{amb}}-p_{\mathrm{v}}}{\rho}\right)^{\frac{1}{2}}.$$

Dividing the initial radius of the bubble by this quantity, we obtain the collapse timescale

$$\tau_{\mathrm{coll}} = R_0\left(\frac{\rho}{p_{\mathrm{amb}}-p_{\mathrm{v}}}\right)^{\frac{1}{2}}. \tag{5.2}$$

The classical result, obtained by integrating (5.1), predicts that the total collapse ($R=0$) will occur at the moment $0.915\tau_{\mathrm{coll}}$ (see e.g. Plesset & Prosperetti 1977; Batchelor 1967, §6.12).

The ratio of the relaxation time of the macromolecule to the collapse timescale, $\tau_1/\tau_{\mathrm{coll}}$, or, more precisely, the quantity

$$\alpha \equiv \frac{1}{E_{\mathrm{cs}}\tau_{\mathrm{coll}}} = \frac{1}{E_{\mathrm{cs}}R_0}\left(\frac{p_{\mathrm{amb}}-p_{\mathrm{v}}}{\rho}\right)^{\frac{1}{2}},$$

will play an important role in the subsequent analysis. Note, however, that the significance of this quantity is quite different from that of its counterpart (4.3) in the growth case: in the latter

$$\frac{1}{R_0}\left(\frac{p_{\mathrm{v}}-p_{\mathrm{amb}}}{\rho}\right)^{\frac{1}{2}}$$

provided an estimate of the maximum rate of strain reached during the entire growth process, while here $\tau_{\mathrm{coll}}^{-1}$ provides an estimate of the rate-of-strain values reached soon after the beginning of collapse, the rate-of-strain magnitude becoming very large as the collapse proceeds and $R\to 0$.

We can now proceed to the calculation of the polymer-induced stress. We have for the last term in (2.1), neglecting the Newtonian contribution,

$$3\int_R^\infty \frac{\tau_{rr}}{r}\,\mathrm{d}r = 6\eta_{\mathrm{s}}\int_R^\infty \frac{\zeta_+ E}{r}\,\mathrm{d}r = -12\kappa\eta_{\mathrm{s}}R^2\dot{R}\int_R^\infty \frac{\xi^3}{r^4}\,\mathrm{d}r, \tag{5.3}$$

where ξ is the relative elongation of a material line element, lying along the radial direction, from the critical point where E was equal to E_{cs}. Denoting the position of this point as r_{cs}, we have

$$\xi = \left(\frac{r_{\text{cs}}}{r}\right)^2, \tag{5.4}$$

since as a thin spherical shell of fluid contracts towards the origin $r = 0$, its area decreases as r^2, and thus its thickness must increase as r^{-2}. Thus the last integral in (5.3) is

$$\int_R^\infty \frac{\xi^3}{r^4}\,\mathrm{d}r = \int_R^\infty \frac{r_{\text{cs}}^6}{r^{10}}\,\mathrm{d}r. \tag{5.5}$$

The value of r_{cs} is different for different fluid elements (labelled by their position r_0 at time zero); thus, generally speaking, r_{cs} is a function of both r (i.e. the current position of the fluid element) and time. To make the problem manageable, we will have to introduce some approximations.

5.1. *Calculation of the position of the critical point*

Consider the bubble collapse from time zero. In the beginning the rates of strain are too small to cause the CS events to occur, and so the bubble follows the classical solution (5.1). At any given moment the highest strain-rate value is at the surface of the bubble (see (3.1)), and so the first CS events occur when this value becomes equal to E_{cs}, i.e.

$$-\frac{2\dot{R}}{R_{\text{cs}}} = E_{\text{cs}}$$

or

$$R_{\text{cs}} = \frac{2}{E_{\text{cs}}}\left(\frac{2}{3}\frac{p_{\text{amb}}-p_{\text{v}}}{\rho}\right)^{\frac{1}{2}}\left[\left(\frac{R_0}{R_{\text{cs}}}\right)^3-1\right]^{\frac{1}{2}},$$

which can be rewritten as

$$\frac{R_{\text{cs}}}{R_0} = 1.63\alpha\left[\left(\frac{R_0}{R_{\text{cs}}}\right)^3-1\right]^{\frac{1}{2}}. \tag{5.6}$$

For large α we obtain $R_{\text{cs}} \approx R_0$ (i.e. E_{cs} is reached on the surface almost immediately), while for small α and thus $R_{\text{cs}} \ll R_0$, we get (keeping two significant digits only)

$$\frac{R_{\text{cs}}}{R_0} \approx 1.2\alpha^{\frac{2}{5}}.$$

Consider the case of large α first. We see that $r_{\text{cs}} \approx R_0$ for the fluid elements near the bubble surface. For the fluid elements far from the bubble surface r_{cs} is, of course, different; however, for these fluid elements the ratio r/r_0 is never much different from 1, and so the polymer chains in these fluid elements can never reach significant elongations. To see this most clearly, recall the relation (4.7),

$$r^3-R^3 = r_0^3-R_0^3,$$

which expresses the conservation of mass within a spherical material surface of initial radius r_0, and consider the case $r_0 \gg R_0$. Indeed, only those polymer chains that are initially near the bubble surface will be stretched strongly enough to produce significant stress: we can show, using (4.7), that in order to ever reach elongation $\xi > 5$ (which would be barely enough to produce an observable polymer effect in a dilute solution), the polymer chain must reside in a fluid element with $r_0 < 1.03R_0$, i.e. at the start of the collapse it must reside in the immediate vicinity of the bubble surface. We have quite generally (for any α)

$$\xi = \left(\frac{r_{cs}}{r}\right)^2 < \left(\frac{r_0}{r}\right)^2 < \frac{r_0^2}{(r_0^3-R_0^3)^{\frac{2}{3}}} = \left[1-\left(\frac{R_0}{r_0}\right)^3\right]^{-\frac{2}{3}},$$

and therefore sufficiently high elongations $\xi > \xi_{\min}$ are possible only for the fluid elements with

$$\frac{r_0}{R_0} < 1+\frac{1}{3\xi_{\min}^{\frac{3}{2}}};$$

the right-hand side of the last relation being 1.03 for $\xi_{\min} = 5$.

Hence for large α a good approximation will be

$$r_{cs} = R_{cs} = R_0$$

for all those polymer chains that can contribute significantly to the effect.

Consider now the case of small α. The first CS events occur relatively late, when $R = R_{cs} = 1.2\alpha^{\frac{2}{5}}R_0$. The subsequent CS events occur when the following relation (obtained using (3.1) and (5.1) for $R \ll R_0$) is satisfied:

$$\left(\frac{r_{cs}}{R_0}\right)^3 = 1.63\alpha\left(\frac{R}{R_0}\right)^{\frac{1}{2}}. \tag{5.7}$$

Since R in this relation will be less than R_{cs}, it is clear that $r_{cs} < R_{cs}$ also (recall that R_{cs} can be obtained from the same equation (5.7) applied at the surface of the bubble). However, we can show that for those polymer chains that can contribute significantly to the effect, the difference between r_{cs} and R_{cs} is negligible.

Using (5.4) and (4.7) and taking into account that $r_{cs} < R_{cs}$, we obtain

$$\xi < \frac{r_{cs}^2}{(r_0^3-R_0^3)^{\frac{2}{3}}} < \frac{R_{cs}^2}{(r_0^3-R_0^3)^{\frac{2}{3}}}$$

and thus

$$\xi < (1.63\alpha)^{\frac{4}{5}}\left[\left(\frac{r_0}{R_0}\right)^3-1\right]^{-\frac{2}{3}}.$$

It is then an easy matter to show that sufficiently high elongations $\xi > \xi_{\min}$ are possible only for the fluid elements with

$$\frac{r_0}{R_0} < 1+\frac{0.60\alpha^{\frac{6}{5}}}{\xi_{\min}^{\frac{3}{2}}};$$

the right-hand side of the last relation being $1+0.054\alpha^{\frac{6}{5}}$ for $\xi_{\min} = 5$.

Now we can find r_{cs} for the fluid element with the label $r_0 = (1+0.054\alpha^{\frac{6}{5}})R_0$, using (5.7) and (4.7). The result is that this r_{cs} is less than R_{cs} by only about 1 %. This also means that R at the moment when the CS event occurs in this fluid element is only slightly less than R_{cs}, and so the polymer chains that underwent the CS before are only slightly elongated. This provides an *a posteriori* justification for using the inviscid-fluid kinematics of (5.1) in deriving (5.7).

Thus, a good approximation for small α will be

$$r_{cs} = R_{cs} = 1.2\alpha^{\frac{2}{5}}R_0,$$

again for all those polymer chains that can contribute significantly to the effect.

Observe that we found the approximation $r_{cs} = R_{cs}$ to be acceptable for both large and small α. It is then reasonable to assume that it will be acceptable also for $\alpha = O(1)$. In this case R_{cs}/R_0 must be found by solving (5.6) numerically (say, by Newton's method) for each given α. Quite possibly, a good interpolation formula for R_{cs}/R_0 as a function of α could be devised. We shall not have a need to do this in the present work.

5.2. *The final form of the basic equation*

We can now calculate the integral (5.5). The upper limit in this integral should, in principle, define the extent of the fluid where the polymer chains are stretched ($\xi > 1$). However, when the polymer effect is significant, this value is at least a few times larger than the lower limit R, which means, in view of the very strong dependence on r of the integrand, that we can safely extend the integration to infinity, with the error thus introduced being small.

This very strong dependence on r means also that hardly any error is introduced by using the above approximate values for r_{cs}: most of the effect will be determined by those polymer chains that are close to the bubble, and for these chains the above expressions for r_{cs} are quite accurate.

We obtain in the case of small α

$$\int_R^\infty \frac{r_{\mathrm{cs}}^6}{r^{10}}\,\mathrm{d}r = 0.33\alpha^{\frac{12}{5}}\frac{R_0^6}{R^9},$$

while for large α we have
$$\int_R^\infty \frac{r_{\mathrm{cs}}^6}{r^{10}}\,\mathrm{d}r = \frac{1}{9}\frac{R_0^6}{R^9}.$$

The equation describing the bubble collapse finally takes the form (see (2.1) and (5.3))

$$\rho(R\ddot{R}+\tfrac{3}{2}\dot{R}^2) = -(p_{\mathrm{amb}}-p_{\mathrm{v}})-4\psi\kappa\eta_{\mathrm{s}}R_0^6\frac{\dot{R}}{R^7}, \tag{5.8}$$

where
$$\psi \equiv \begin{cases} \alpha^{\frac{12}{5}} & \text{for small } \alpha \\ \frac{1}{3} & \text{for large } \alpha \end{cases}.$$

Note that the two expressions for ψ have a crossover point $\alpha = 0.63$.

Comparing with (2.2), we observe that the ratio of the polymer term to the Newtonian viscous term is

$$\psi\kappa\left(\frac{R_0}{R}\right)^6.$$

Let us now put (5.8) in dimensionless form. Using R_0 as a lengthscale and τ_{coll} as a timescale, we obtain (denoting the dimensionless radius of the bubble as $\mathscr{R}$; the dot will now mean differentiation with respect to the dimensionless time t)

$$\mathscr{R}\ddot{\mathscr{R}}+\tfrac{3}{2}\dot{\mathscr{R}}^2 = -1-\lambda\frac{\dot{\mathscr{R}}}{\mathscr{R}^7}, \tag{5.9}$$

where
$$\lambda \equiv \frac{4\psi\kappa\eta_{\mathrm{s}}}{\tau_{\mathrm{coll}}(p_{\mathrm{amb}}-p_{\mathrm{v}})} = \frac{4\psi\kappa\eta_{\mathrm{s}}}{R_0\rho^{\frac{1}{2}}(p_{\mathrm{amb}}-p_{\mathrm{v}})^{\frac{1}{2}}} \approx \frac{1.2\psi c[\eta]\,\eta_{\mathrm{s}}}{R_0\rho^{\frac{1}{2}}(p_{\mathrm{amb}}-p_{\mathrm{v}})^{\frac{1}{2}}}. \tag{5.10}$$

The parameters α and λ obviously determine the importance of the polymer effect. Let us look at some numbers. Take $p_{\mathrm{amb}}-p_{\mathrm{v}} = 1$ bar; $\rho = 1\ \mathrm{g/cm^3}$; $\tau_1 = 3.5\times10^{-3}$ s (the last value corresponding, for example, to Polyox with molecular weight $M = 8\times 10^6$ g/mol in water, see Ryskin 1987*a*). Then, assuming that $E_{cs} \approx \tau_1^{-1}$, we obtain

$$\alpha \approx \frac{3.5}{R_0},$$

where R_0 is in centimetres.

This obviously corresponds to the large-α case in most situations of interest, so that

$$\lambda \approx \frac{0.4c[\eta]\,\eta_s}{R_0\,\rho^{\frac{1}{2}}(p_{amb}-p_v)^{\frac{1}{2}}} = \frac{0.4c[\eta]\,\eta_s}{\rho R_0\, U}. \tag{5.11}$$

In dilute solutions $c[\eta] \ll 1$. Consider a typical case: $R_0 = 1$ mm, $p_{amb}-p_v = 1$ bar, $\rho = 1$ g/cm^3, $\eta_s = 0.01$ g cm^{-1} s^{-1}, $c[\eta] = 0.05$ (the last value corresponding, for example, to the 20 p.p.m. solution of Polyox mentioned earlier). Then we have $\lambda = 2\times 10^{-6}$.

To estimate an upper limit on λ, it is helpful to consider first the case of a cavitation bubble in a Newtonian solvent alone. We have in dimensionless form

$$\mathscr{R}\ddot{\mathscr{R}}+\tfrac{3}{2}\dot{\mathscr{R}}^2 = -1-\mu\frac{\dot{\mathscr{R}}}{\mathscr{R}}, \tag{5.12}$$

where
$$\mu \equiv \frac{4\eta_s}{R_0\,\rho^{\frac{1}{2}}(p_{amb}-p_v)^{\frac{1}{2}}} = \frac{4\eta_s}{\rho R_0\, U}$$

is, essentially, an inverse of the Reynolds number. For a cavitation-type collapse to occur at all, μ must be less than some critical value, otherwise the bubble diminishes in size gradually, $\dot{\mathscr{R}}$ being $O(1)$ or less throughout the process. The critical value of μ is about 0.5 (I found this value by integrating numerically (5.12) with different μ, choosing the latter according to the bisection algorithm).

Since $\lambda = 0.1c[\eta]\,\mu$, we see that $\lambda < 0.05c[\eta]$ (otherwise no cavitation would occur even in the absence of polymer). Clearly, λ is unlikely to exceed $O(10^{-2})$ in any situation of interest. The values of λ encountered in experiment are likely to lie in the 10^{-6}–10^{-4} range. Nevertheless, the polymer term in (5.9) will play a significant role in the dynamics of the bubble collapse, in view of its very fast growth when $\mathscr{R}\to 0$.

The different terms in (5.9) have obvious physical meaning: the left-hand side describes the inertia of the accelerating liquid, the -1 term on the right represents the pressure difference $p_{amb}-p_v$, responsible for the initiation of the collapse, and the last term gives the polymer effect. It is useful to distinguish four consecutive stages of the collapse, and this is done below. But let us first write down, for reference, the form taken by the classical inviscid-fluid solution (i.e. with $\lambda = 0$) in the dimensionless variables:

$$\dot{\mathscr{R}} = -\left(\frac{2}{3}\right)^{\frac{1}{2}}\left(\frac{1}{\mathscr{R}^3}-1\right)^{\frac{1}{2}}, \tag{5.13}$$

$$\ddot{\mathscr{R}} = -\frac{1}{\mathscr{R}^4}. \tag{5.14}$$

Now the phases of the collapse are:

(i) Phase I ('speed-up'): beginning from rest, the liquid is accelerated by the pressure difference. Initially,

$$\mathscr{R}\approx 1;\quad \dot{\mathscr{R}}\approx 0;\quad \ddot{\mathscr{R}}\approx -1;$$

the magnitude of the bubble wall velocity then grows roughly linearly with time. The polymer effect is negligible, and the inertial forces are balanced by the pressure forces. The radius of the bubble decreases substantially during this phase.

(ii) Phase II ('collapse'):

$$\mathscr{R} \ll 1; \quad \dot{\mathscr{R}} = -\left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{1}{\mathscr{R}^{\frac{3}{2}}}; \quad \tfrac{3}{2}\dot{\mathscr{R}}^2 \approx \frac{1}{\mathscr{R}^3}.$$

The two terms on the left-hand side of (5.9) are both very large and essentially balance each other, with both the pressure difference effect and the polymer effect being negligible. This is essentially a purely inertial motion; however, since the volume (and thus the mass) of the fast moving liquid in the neighbourhood of the bubble is shrinking, its velocity is rapidly increasing (see Batchelor 1967, p. 489).

Phase I and Phase II are described by the classical solution (5.13), (5.14) and are the only ones to occur in the absence of polymer.

(iii) Phase III ('slowdown by polymer'): $\mathscr{R}$ becomes so small that the polymer term becomes comparable with the inertial terms on the left-hand side, and the magnitude of $\dot{\mathscr{R}}$ goes through a maximum and then falls rapidly; the pressure forces are still small in comparison with either inertial or polymer forces, which balance each other.

(iv) Phase IV ('polymer-controlled creep'): $\dot{\mathscr{R}}$ became so small that the inertial terms are negligible; the pressure forces are balanced by the polymer forces, and the bubble radius slowly decreases.

5.3. *Approximate analytical solution*

Our task is now to find the approximate form of the solution for Phases III and IV. That for Phase IV is easy:

$$\dot{\mathscr{R}} = -\frac{1}{\lambda}\mathscr{R}^7, \tag{5.15}$$

so if Phase IV begins at $t = t_{\mathrm{IV}}$ when $\mathscr{R} = \mathscr{R}_{\mathrm{IV}}$, we have from that moment on

$$\frac{1}{\mathscr{R}^6} = \frac{1}{\mathscr{R}_{\mathrm{IV}}^6} + \frac{6}{\lambda}(t - t_{\mathrm{IV}}).$$

The last expression applies only while the polymer chains are stretched, whereas (5.15) shows that E will approach 0 as $\mathscr{R}^6$, and so at some point the chains will begin to curl back. We shall not consider these phenomena here.

Phase III is described by

$$\mathscr{R}\ddot{\mathscr{R}} + \tfrac{3}{2}\dot{\mathscr{R}}^2 = -\lambda\frac{\dot{\mathscr{R}}}{\mathscr{R}^7}. \tag{5.16}$$

This equation can be integrated once if $\mathscr{R}$ is taken as a new independent variable, and $\mathscr{R}^3\dot{\mathscr{R}}^2$ as a new dependent variable. The result is

$$\dot{\mathscr{R}} = \frac{\text{const}}{\mathscr{R}^{\frac{3}{2}}} + \tfrac{2}{11}\lambda\frac{1}{\mathscr{R}^7},$$

where const is an arbitrary constant.

For small λ this can be matched to the Phase II solution, giving

$$\dot{\mathscr{R}} = -\left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{1}{\mathscr{R}^{\frac{3}{2}}} + \tfrac{2}{11}\lambda\frac{1}{\mathscr{R}^7}. \tag{5.17}$$

Equation (5.17) thus provides a good approximation to the exact solution in Phases II and III, up to the moment when $\mathscr{R}$ becomes so small that the neglect of the pressure term, which led to (5.16), becomes unjustifiable, i.e. the motion enters Phase IV. We can switch over from $\dot{\mathscr{R}}$ given by (5.17) to the 'creep' solution (5.15) at the moment when they become equal. The corresponding value of $\mathscr{R}$, denoted $\mathscr{R}_{\text{IV}}$, should be found by equating the right-hand sides of (5.15) and (5.17); however, (5.15) is much smaller than each of the two terms in (5.17), and so an approximate value of $\mathscr{R}_{\text{IV}}$ can be found by equating the right-hand side of (5.17) to zero. We thus obtain

$$\mathscr{R}_{\text{IV}} \approx 0.76\lambda^{\frac{2}{11}}, \tag{5.18}$$

and the value of $\dot{\mathscr{R}}$ at the moment of switchover is then found from (5.15) as

$$\dot{\mathscr{R}} \approx -0.15\lambda^{\frac{3}{11}}. \tag{5.19}$$

The approximate solution to the problem of bubble collapse is thus given by the classical inviscid-fluid solution (5.13) in the very beginning, then by the expression (5.17) till the magnitude of $\dot{\mathscr{R}}$ falls down to the value (5.19), and after that by the expression (5.15).

The absolute value of the bubble wall velocity $\dot{\mathscr{R}}$ increases without limit in the classical solution (5.13), but has a maximum if polymer is present. It is easy to show using (5.17) that the maximum value of $|\dot{\mathscr{R}}|$ is

$$|\dot{\mathscr{R}}|_{\max} \approx 0.63\lambda^{-\frac{3}{11}} \tag{5.20}$$

and that this maximum is reached when

$$\mathscr{R} \approx \lambda^{\frac{2}{11}}. \tag{5.21}$$

That is, between $\mathscr{R} \approx \lambda^{\frac{2}{11}}$ and $\mathscr{R} \approx 0.76\lambda^{\frac{2}{11}}$ the magnitude of the bubble wall velocity drops from $0.63\lambda^{-\frac{3}{11}}$ to $0.15\lambda^{\frac{3}{11}}$. At small λ this means that the collapse is abruptly arrested after the velocity has reached its peak; the phenomena associated with the final stage of the collapse (strong sound emission, luminescence, etc.) will not occur during the very slow Phase IV.

Note that even for λ as small as 10^{-7} the arrest occurs at $\mathscr{R}_{\text{IV}} \approx 0.04$ – not a negligible fraction of the initial size.

It might seem at a first glance that the polymer effect is roughly equivalent to a viscosity increase, and one might wonder if similar effects occur in a Newtonian fluid. This is not the case, and it is instructive to see why.

In a Newtonian fluid the collapse is described by (5.12), where the parameter μ is, typically, a small number (for $R_0 = 1$ mm in water, with $p_{\text{amb}} - p_{\text{v}} = 1$ bar, we have $\mu = 4 \times 10^{-4}$). The viscous term then has negligible influence in the beginning (Phases I and II). Consider now an analogue of Phase III where the pressure term -1 can be neglected, while the viscous effect may become important. Integrating (5.12) once with -1 dropped, and matching the result to the Phase II solution, we obtain

$$\dot{\mathscr{R}} = -\left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{1}{\mathscr{R}^{\frac{3}{2}}} - \frac{2\mu}{\mathscr{R}}. \tag{5.22}$$

That is, the viscous effect appears to *accelerate* the collapse! This paradoxical conclusion becomes more palatable if we recall that (5.22) gives $\dot{\mathscr{R}}$ as a function of $\mathscr{R}$ and not of time; during the initial start-up (i.e. Phase I) the viscous effect is certainly decelerating (see (5.12)), and so while (5.22) predicts a higher magnitude of $\dot{\mathscr{R}}$ for a given $\mathscr{R}$ than the inviscid-fluid solution (5.13), this $\mathscr{R}$ is reached later.

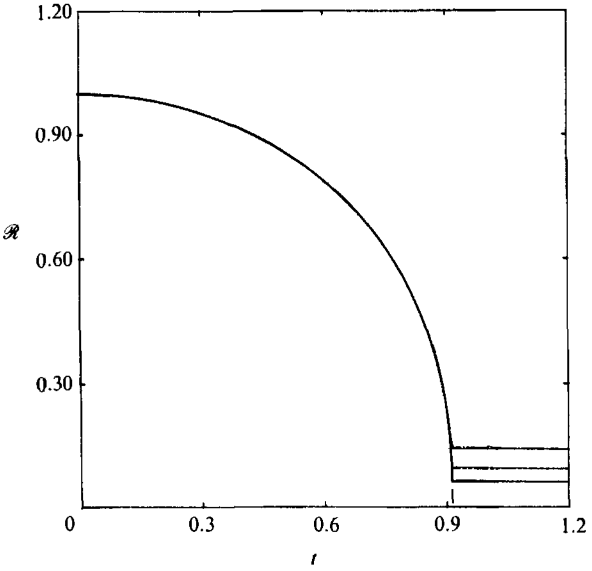

FIGURE 1. Bubble radius *vs.* time. The lengthscale is the initial bubble radius R_0, and the timescale is τ_{coll}, defined in (5.2): ---, $\lambda = 0$ (inviscid fluid); ——, $\lambda = 10^{-6}$; ----------, $\lambda = 10^{-5}$; —·—·—, $\lambda = 10^{-4}$.

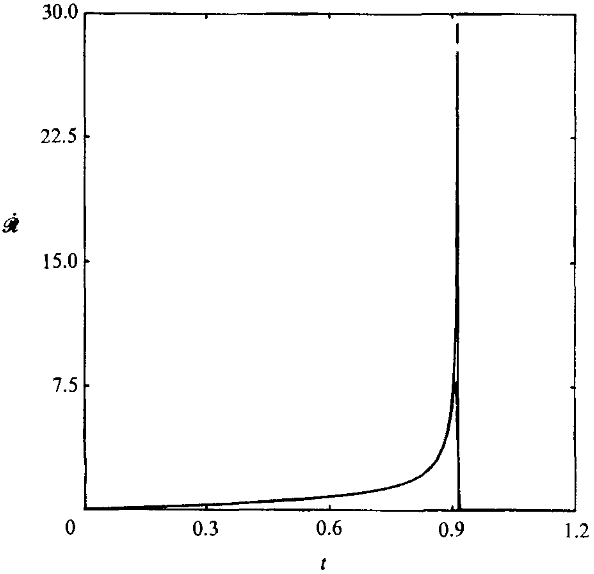

FIGURE 2. Bubble wall velocity *vs.* time. For symbols see figure 1.

The most important conclusion that can be drawn from (5.22) is that the Newtonian viscous effects cannot substantially retard the collapse. The relative magnitude of the viscous correction to the classical solution is $O(\mu\mathscr{R}^{\frac{1}{2}})$ – very small near the end of the collapse. Nothing similar to the polymer-induced arrest takes place.

Of course, if μ is $O(1)$ or higher, the motion will be retarded very significantly, so that $\dot{\mathscr{R}}$ is never greater than $O(1)$. But then we are not dealing with cavitation.

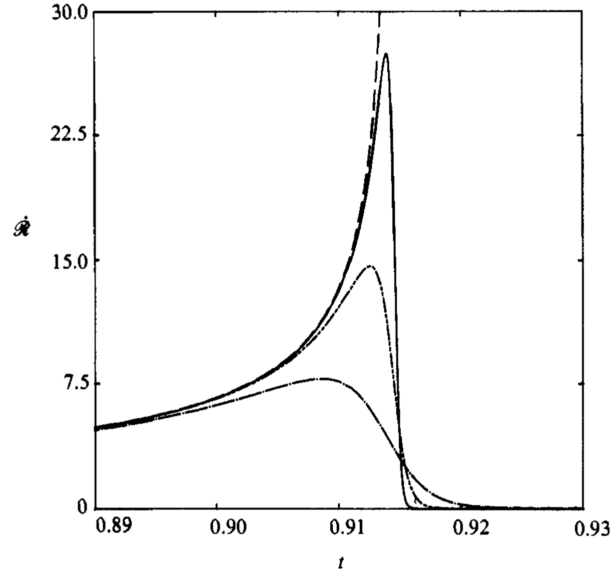

FIGURE 3. A portion of figure 2, blown-up in time.

5.4. *Numerical solution*

Figures 1, 2, 3 present the results of numerical integration of (5.9) for cases $\lambda = 10^{-6}$; 10^{-5}; 10^{-4}. The classical inviscid-fluid solution ($\lambda = 0$) is also shown for comparison.

It can be seen that the qualitative description of the solution in terms of four distinct phases is confirmed by the computation. The approximate results (5.18), (5.19), (5.20), (5.21) are found to agree very well with the corresponding values obtained numerically, the error being within about 1% for λ in the range 10^{-7} to 10^{-2}.

6. Polymer effect on the sound emission during the collapse

A collapsing bubble radiates sound as a simple (monopole) source, whose strength $\dot{q}$ is a function of time, given by (Lighthill 1978, pp. 19, 33)

$$\dot{q} = \rho\ddot{V} = 4\pi\rho(2R\dot{R}^2 + R^2\ddot{R}),$$

where V is the volume of the bubble. Knowing $\dot{q}$, one obtains the sound (excess) pressure at distance r as

$$\frac{\dot{q}_{\mathrm{ret}}}{4\pi r}.$$

Here $\dot{q}_{\mathrm{ret}}$ is the value of $\dot{q}$ at an earlier moment, back in time by r/c_{ac}, where c_{ac} is the speed of sound.

Introducing the dimensionless source strength $\dot{\mathcal{Q}}$ as

$$\dot{\mathcal{Q}} \equiv \frac{\dot{q}}{4\pi R_0(p_{\mathrm{amb}} - p_{\mathrm{v}})}, \tag{6.1}$$

we have

$$\dot{\mathcal{Q}} = 2\mathcal{R}\dot{\mathcal{R}}^2 + \mathcal{R}^2\ddot{\mathcal{R}}.$$

Figure 4 presents $\dot{\mathcal{Q}}$ *vs.* time, obtained numerically for $\lambda = 10^{-6}$; 10^{-5}; 10^{-4}.

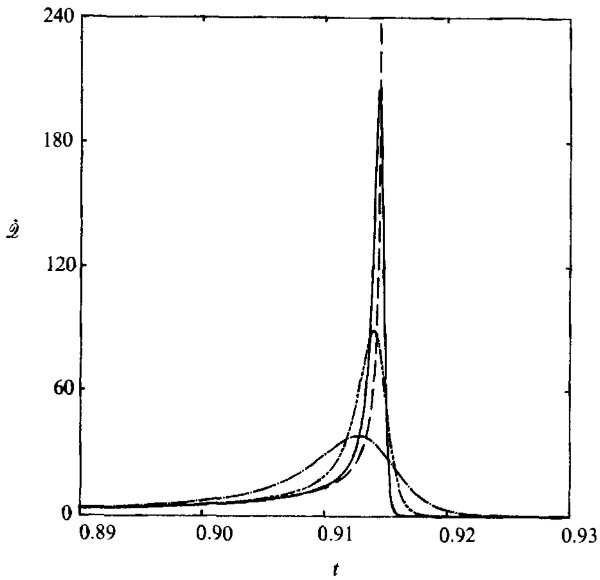

FIGURE 4. Strength of the acoustic monopole due to a collapsing bubble *vs.* time. For symbols see (6.1) and figure 1.

Using (5.17), one finds an estimate for the maximum value of $\dot{\mathscr{Q}}$ as $1.23\lambda^{-\frac{4}{11}}$ and for $\mathscr{R}$ at the moment when this maximum is reached as $0.81\lambda^{\frac{2}{11}}$. Comparison with the numerical solutions for λ in the range 10^{-7} to 10^{-2} shows, however, that the above numerical factors are somewhat underestimated; the numerical results are very well represented by the relations

$$\dot{\mathscr{Q}}_{\max} \approx 1.34\lambda^{-\frac{4}{11}} \qquad \text{at} \qquad \mathscr{R} \approx 0.85\lambda^{\frac{2}{11}}.$$

The acoustic power output is given by (Lighthill 1978, p. 22)

$$\frac{\dot{q}^2}{4\pi\rho c_{\mathrm{ac}}};$$

integration of this quantity over the time of collapse gives the total acoustic energy radiated:

$$W_{\mathrm{ac}} = 4\pi\frac{R_0^3(p_{\mathrm{amb}}-p_{\mathrm{v}})^{\frac{3}{2}}}{c_{\mathrm{ac}}\rho^{\frac{1}{2}}}\int_1^0 \frac{\dot{\mathscr{Q}}^2}{\dot{\mathscr{R}}}\,\mathrm{d}\mathscr{R}.$$

The ratio of W_{ac} to the potential energy of the bubble available before the collapse, $\frac{4}{3}\pi R_0^3(p_{\mathrm{amb}}-p_{\mathrm{v}})$, is

$$\frac{W_{\mathrm{ac}}}{W_{\mathrm{pot}}} = 3\frac{(p_{\mathrm{amb}}-p_{\mathrm{v}})^{\frac{1}{2}}}{c_{\mathrm{ac}}\rho^{\frac{1}{2}}}\int_1^0 \frac{\dot{\mathscr{Q}}^2}{\dot{\mathscr{R}}}\,\mathrm{d}\mathscr{R} = 3\frac{U}{c_{\mathrm{ac}}}\int_1^0 \frac{\dot{\mathscr{Q}}^2}{\dot{\mathscr{R}}}\,\mathrm{d}\mathscr{R}. \tag{6.2}$$

This ratio must certainly be less than one; in the studies of cavitation noise it is often assumed that it may approach one closely (Ross 1987, p. 218; Baiter 1982). Our situation is different: the polymer effect is dissipative in nature, and a straightforward calculation using (5.17) shows that all the potential energy is dissipated into heat by the end of Phase III. While approximate (some dissipation occurs, of course, during Phase IV, but this correction is $O(\lambda^{\frac{6}{11}})$), this result is fully consistent with the

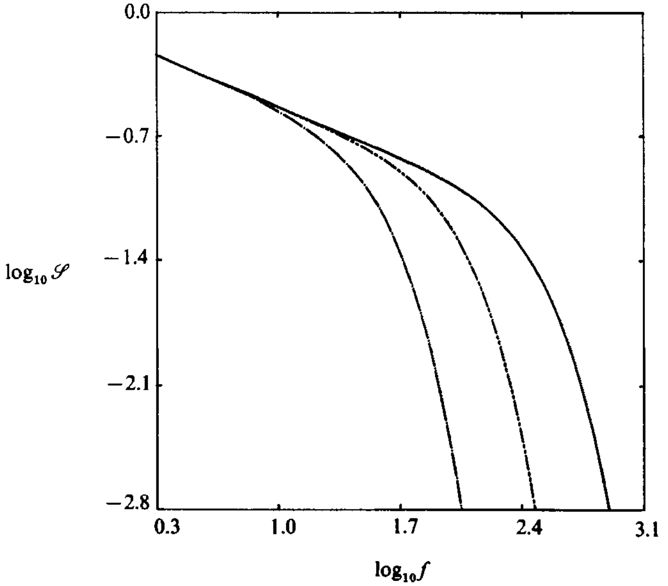

FIGURE 5. Polymer effect on the spectrum of acoustic radiation from a collapsing bubble. For symbols see (6.3) and figure 1.

approximations that led to (5.17), (5.18), and thus provides a useful check on the latter. But this means that the ratio (6.2) must be a rather small number, otherwise one has to conclude that some of the basic assumptions of the present theory are violated, and a new analysis, taking into account the compressibility of the liquid (see Prosperetti 1987), is required. For example, for water and $p_{\mathrm{amb}} - p_{\mathrm{v}} = 1$ bar, the factor in front of the integral in (6.2) is about 0.02, so the integral must not exceed $O(10)$.

Using (5.17), one obtains an estimate for this integral as $0.95\lambda^{-\frac{3}{11}}$; however, the numerical results are well represented by a somewhat different expression, namely

$$\int_1^0 \frac{\dot{\mathscr{Q}}^2}{\mathscr{R}} \mathrm{d}\mathscr{R} \approx 0.86\lambda^{-\frac{3}{11}}.$$

Of primary interest to an experimentalist is the spectrum of the sound emitted by a collapsing bubble. Let us define the one-sided power spectral density $\mathscr{S}(f)$ as

$$\mathscr{S}(f) = 2|\mathscr{G}(f)|^2, \tag{6.3}$$

where f is the dimensionless frequency (the scale for frequency being $\tau_{\mathrm{coll}}^{-1}$), and $\mathscr{G}(f)$ is the Fourier transform of $\dot{\mathscr{Q}}(t)$, i.e.

$$\mathscr{G}(f) = \int_{-\infty}^{\infty} \dot{\mathscr{Q}}(t)\, \mathrm{e}^{2\pi i f t} \mathrm{d}t.$$

$\mathscr{S}(f)$ was computed using the discrete Fourier transform with subsequent smoothing (see Press *et al.* 1986, p. 423). Figure 5 presents the results for $\lambda = 10^{-6}$; 10^{-5}; 10^{-4} (only a part of the spectrum is shown since low frequencies are not affected by the polymer). The common part of the curves has the slope -0.4 and is associated with the Phase II velocity–radius relationship

$$\dot{\mathscr{R}} \propto -\mathscr{R}^{-\frac{3}{2}},$$

see Fitzpatrick & Strasberg (1957, p. 253), Blake (1986, p. 408). Without the polymer the spectrum would continue asymptotically with this slope until some other, neglected here, effects became important (non-condensable gas, compressibility, etc.; see Blake 1986).

The presence of the polymer results in a rather sharp cutoff of the spectrum. The cutoff frequency f_c, by the bandwidth theorem (see e.g. Champeney 1973), should scale as the inverse of the duration of the pulse in the acoustic pressure near the end of the collapse. This duration can be roughly estimated as the change in $\mathscr{R}$ during the pulse (which scales as $\lambda^{\frac{2}{11}}$), divided by the corresponding velocity (which scales as $\lambda^{-\frac{3}{11}}$). We thus conclude that f_c should scale as $\lambda^{-\frac{5}{11}}$.

The numerical results are in agreement with this scaling. If we define f_c as a frequency such that $\mathscr{S}(f_c)$ is one-half of what it would be in the absence of polymer, i.e. of what $f^{-0.4}$ asymptote would yield at f_c, we find that the numerical results are described by

$$f_c \approx 0.52\lambda^{-\frac{5}{11}},$$

with an error of about 1%.

The above discussion is concerned with the sound emitted by a single bubble. It should also be relevant, however, to the phenomenon of cavitation noise, if the bubbles' sizes can be considered uniform: the spectrum of a random sequence of identical signals (the *shot noise*) is essentially the same as that of a signal itself (see e.g. Champeney 1973).

7. Discussion

To summarize, we found that the polymer additive does not affect the growth of a cavitation bubble, and affects the collapse only near its end. The most interesting effects of cavitation do happen near the end of the collapse, however, and therefore the polymer effect may be quite important.

The polymer arrests the collapse when the dimensionless bubble radius is about $\lambda^{\frac{2}{11}}$ (where λ is defined by (5.10) or (5.11)) and imposes an upper limit of about $\lambda^{-\frac{3}{11}}$ on the bubble wall velocity, and of about $\lambda^{-\frac{4}{11}}$ on the acoustic source strength. The polymer also sharply curtails the high-frequency part of the acoustic radiation spectrum, with the cutoff frequency being of order $\lambda^{-\frac{5}{11}}$.

Only those polymer chains that reside initially in the immediate vicinity of the bubble surface (within a layer whose thickness is at most about 3% of the initial bubble radius) are responsible for these effects; the chains initially outside of this layer never reach the high elongations necessary to produce large stresses, and, in fact, might as well be absent.

Unfortunately, I am not aware of experimental data that could be directly compared to the present results. Ting & Ellis (1974) observed that the growth of a bubble was unaffected by the polymer, but they used gas bubbles, as did Shima, Tomita & Ohno (1984), who observed shortening (!) of the collapse upon addition of polymer. Spark-generated vapour bubbles were used by Chahine & Fruman (1979), while Kezios & Schowalter (1986) produced vapour bubbles by focusing a laser beam into a small volume of fluid. In both cases, no effect of polymer additive on the dynamics of spherical bubbles was observed. The maximum radius of the bubbles was, however, relatively large, and so λ could be too small for the effect to be detectable without using the most advanced optical techniques (Lauterborn & Vogel 1984; Lauterborn & Hentschel 1985).

On the other hand, Hoyt (1977) and Crum & Brosey (1984) observed inhibition of *acoustic* cavitation by polymer additives. While this is more likely to depend on single-bubble dynamics than the similar effect in a flow past a body, etc. (see § 1), the phenomenon as a whole is highly complex and requires separate investigation. Crum & Brosey (1984) proposed that the polymer effect is due entirely to changes in the surface tension on the vapour–liquid interface, with resulting changes in the nucleation process. This explanation is unlikely to hold, however, for the fibre suspensions, that were observed to inhibit acoustic cavitation also (Hoyt 1977).

Measurements of the cavitation noise are equally inconclusive: while Reitzer *et al.* (1985) observed a reduction in the acoustic pressure upon addition of polymer to a cavitating flow past a cylinder, in the ultrasonic cavitation damage experiments of Tsujino (1987) an increase in the acoustic pressure was observed. In both cases the overall phenomena are far too complicated for a direct comparison with the single-bubble results. The same is true of the orifice-flow cavitation experiments by Oba *et al.* (1978), who observed both the reduction in the acoustic pressure and the downward shift in the principal frequency range upon addition of polymer.

It appears that further careful experiments will need to be done before a quantitative check of the present results becomes possible.

If a collapsing bubble contains, in addition to vapour, some amount of a permanent (non-condensable) gas, this gas is strongly compressed, essentially adiabatically, and the resulting temperature and pressure may be so high that free radicals are created. Recombination of these free radicals produces luminescence (often called sonoluminescence since it was initially observed in the experiments where cavitation was generated by high-intensity ultrasound), see, for example, Walton & Reynolds (1984), Suslick & Flint (1987).

The values of the gas temperature and pressure at any moment are, respectively, $T_0\mathscr{R}^{-3(\gamma-1)}$ and $p_{g0}\mathscr{R}^{-3\gamma}$, where the subscript '0' signifies initial conditions, subscript 'g' stands for gas, and γ is the specific-heat ratio. To find the maximum temperature and pressure attained, which in turn determine the intensity (and spectrum ?) of the luminescence, one needs to solve the modified equation (5.9):

$$\mathscr{R}\ddot{\mathscr{R}}+\tfrac{3}{2}\dot{\mathscr{R}}^2 = -1+\omega\frac{1}{\mathscr{R}^{3\gamma}}-\lambda\frac{\dot{\mathscr{R}}}{\mathscr{R}^{7}},$$

where $\omega \equiv p_{g0}/(p_{\mathrm{amb}}-p_{\mathrm{v}})$. This could provide another way of verification of the present theory if the polymer effect on luminescence were measurable.

In the present work the polymer influence on the fluid dynamics of cavitation was considered. It is well known that a very important influence may also be exerted in the opposite direction: the ultrasonic degradation of polymers in solution has been of interest to workers in polymer science since 1939 (see the review by Basedow & Ebert 1977), and the notion that it originates in the scission of macromolecules by the strong elongational flow during collapse of cavitation bubbles was put forward 30 years ago by Thomas (1959). Implications of the present theory for a quantitative analysis of the ultrasonic degradation will be discussed elsewhere. It should be noted, however, that if a significant number of unravelling macromolecules is ruptured in a single bubble collapse, this may reduce the overall polymer effect substantially: the effect of two resulting half-length chains is less than the effect of the original chain by, roughly speaking, a factor of 4, since the polymer-induced stress is proportional to the third power of the instantaneous chain length.

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